Riemann Sums Using Sigma Notation

With sigma notation, a Riemann sum has the convenient compact form

$$f(\bar{x}_1)\,\Delta x + f(\bar{x}_2)\,\Delta x + \cdots + f(\bar{x}_n)\,\Delta x = \sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x.$$

To express left, right, and midpoint Riemann sums in sigma notation, we must identify the points $\bar{x}_k$.

- For left Riemann sums, the left endpoints of the subintervals are $\bar{x}_k = a + (k-1)\,\Delta x$, for $k = 1, \ldots, n$.
- For right Riemann sums, the right endpoints of the subintervals are $\bar{x}_k = a + k\,\Delta x$, for $k = 1, \ldots, n$.
- For midpoint Riemann sums, the midpoints of the subintervals are $\bar{x}_k = a + \left(k - \frac{1}{2}\right)\Delta x$, for $k = 1, \ldots, n$.

The three Riemann sums are written compactly as follows.

DEFINITION Left, Right, and Midpoint Riemann Sums in Sigma Notation

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If $\bar{x}_k$ is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the **Riemann sum** of f on $[a, b]$ is $\sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x$. Three cases arise in practice:

- **left Riemann sum** if $\bar{x}_k = a + (k-1)\,\Delta x$
- **right Riemann sum** if $\bar{x}_k = a + k\,\Delta x$
- **midpoint Riemann sum** if $\bar{x}_k = a + \left(k - \frac{1}{2}\right)\Delta x$, for $k = 1, 2, \ldots, n$

EXAMPLE 5 Calculating Riemann sums Evaluate the left, right, and midpoint Riemann sums of $f(x) = x^3 + 1$ between $a = 0$ and $b = 2$ using $n = 50$ subintervals. Make a conjecture about the exact area of the region under the curve (Figure 5.15).

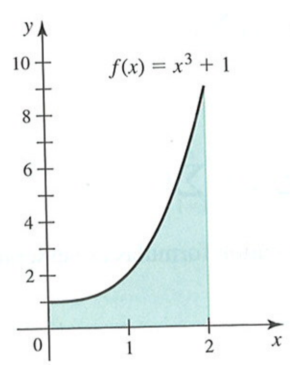


FIGURE 5.15

SOLUTION With $n = 50$, the length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{50} = \frac{1}{25} = 0.04.$$

The value of $\bar{x}_k$ for the left Riemann sum is

$$\bar{x}_k = a + (k-1)\,\Delta x = 0 + 0.04(k-1) = 0.04k - 0.04$$

for $k = 1, 2, \ldots, 50$. Therefore, the left Riemann sum, evaluated with a calculator, is

$$\sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x = \sum_{k=1}^{50} f(0.04k - 0.04)0.04 = 5.8416.$$

To evaluate the right Riemann sum, we let $\bar{x}_k = a + k\,\Delta x = 0.04k$ and find that

$$\sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x = \sum_{k=1}^{50} f(0.04k)0.04 = 6.1616.$$

For the midpoint Riemann sum, we let

$$\bar{x}_k = a + \left(k - \frac{1}{2}\right)\Delta x = 0 + 0.04\left(k - \frac{1}{2}\right) = 0.04k - 0.02.$$

The value of the sum is

$$\sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x = \sum_{k=1}^{50} f(0.04k - 0.02)0.04 \approx 5.9992.$$

Because f is increasing on $[0, 2]$, the left Riemann sum underestimates the area of the shaded region in Figure 5.15, while the right Riemann sum overestimates the area. Therefore, the exact area lies between 5.8416 and 6.1616. The midpoint Riemann sum usually gives the best estimate for increasing or decreasing functions; a reasonable estimate of the area under the curve is 6.

ALTERNATIVE SOLUTION It is worth examining another approach to Example 5 that reappears in Section 5.2. Consider the right Riemann sum given previously:

$$\sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x = \sum_{k=1}^{50} f(0.04k)0.04$$

Rather than evaluating this sum with a calculator, we note that $f(0.04k) = (0.04k)^3 + 1$ and then use the properties of sums:

$$\begin{aligned}
\sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x &= \sum_{k=1}^{50} \underbrace{[(0.04k)^3 + 1]}_{f(\bar{x}_k)}\underbrace{0.04}_{\Delta x} \\
&= \sum_{k=1}^{50} (0.04k)^3\, 0.04 + \sum_{k=1}^{50} 1 \cdot 0.04 \qquad \sum (a_k + b_k) = \sum a_k + \sum b_k \\
&= (0.04)^4 \sum_{k=1}^{50} k^3 + 0.04 \sum_{k=1}^{50} 1 \qquad \sum c a_k = c \sum a_k
\end{aligned}$$

Using the summation formulas for powers of integers in Theorem 5.1, we find that

$$\sum_{k=1}^{50} 1 = 50 \quad \text{and} \quad \sum_{k=1}^{50} k^3 = \frac{50^2 \cdot 51^2}{4}.$$

Substituting the values of these sums into the right Riemann sum, its value is

$$\sum_{k=1}^{50} f(\bar{x}_k)\,\Delta x = \frac{3851}{625} = 6.1616,$$

confirming the result given in the first solution. The idea of evaluating Riemann sums for *arbitrary* values of n is used in Section 5.2, where we evaluate the limit of the Riemann sum as $n \to \infty$.

Related Exercises 35–44

SECTION 5.1 EXERCISES

Review Questions

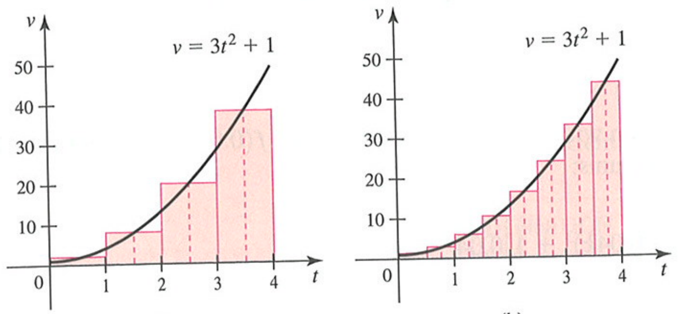
1. Suppose an object moves along a line at 15 m/s for $0 \le t < 2$ and at 25 m/s for $2 \le t \le 5$, where t is measured in seconds. Sketch the graph of the velocity function and find the displacement of the object for $0 \le t \le 5$.
2. Given the graph of the positive velocity of an object moving along a line, what is the geometrical representation of its displacement over a time interval $[a, b]$?
3. Suppose you want to approximate the area of the region bounded by the graph of $f(x) = \cos x$ and the x-axis between $x = 0$ and $x = \pi/2$. Explain a possible strategy.
4. Explain how Riemann sum approximations to the area of a region under a curve change as the number of subintervals increases.
5. Suppose the interval $[1, 3]$ is partitioned into $n = 4$ subintervals. What is the subinterval length Δx? List the grid points x_0, x_1, x_2, x_3, x_4. Which points are used for the left, right, and midpoint Riemann sums?
6. Suppose the interval $[2, 6]$ is partitioned into $n = 4$ subintervals with grid points $x_0 = 2, x_1 = 3, x_2 = 4, x_3 = 5$, and $x_4 = 6$. Write but do not evaluate the left, right, and midpoint Riemann sums for $f(x) = x^2$.
7. Does the right Riemann sum underestimate or overestimate the area of the region under the graph of a positive decreasing function? Explain.
8. Does the left Riemann sum underestimate or overestimate the area of the region under the graph of a positive increasing function? Explain.

Basic Skills

9. Approximating displacement The velocity in ft/s of an object moving along a line is given by $v = 3t^2 + 1$ on the interval $0 \le t \le 4$.

a. Divide the interval $[0, 4]$ into $n = 4$ subintervals, $[0, 1]$, $[1, 2]$, $[2, 3]$, and $[3, 4]$. On each subinterval, assume the object moves at a constant velocity equal to the value of v evaluated at the midpoint of the subinterval and use these approximations to estimate the displacement of the object on $[0, 4]$ (see part (a) of the figure).

b. Repeat part (a) for $n = 8$ subintervals (see part (b) of the figure).

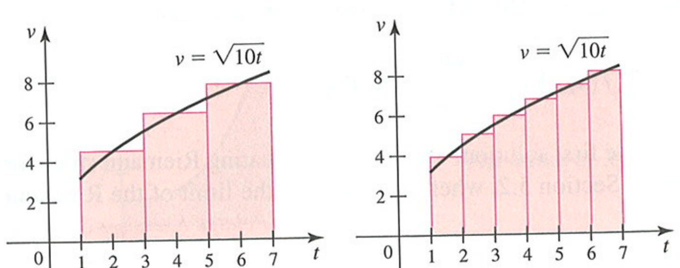


(a) (b)

10. Approximating displacement The velocity in ft/s of an object moving along a line is given by $v = \sqrt{10t}$ on the interval $1 \le t \le 7$.

a. Divide the time interval $[1, 7]$ into $n = 3$ subintervals, $[1, 3]$, $[3, 5]$, and $[5, 7]$. On each subinterval, assume the object moves at a constant velocity equal to the value of v evaluated at the midpoint of the subinterval and use these approximations to estimate the displacement of the object on $[1, 7]$ (see part (a) of the figure).

b. Repeat part (a) for $n = 6$ subintervals (see part (b) of the figure).



(a) (b)

11–14. Approximating displacement *The velocity of an object is given by the following functions on a specified interval. Approximate the displacement of the object on this interval by subdividing the interval into the indicated number of subintervals. Use the left endpoint of each subinterval to compute the height of the rectangles.*

11. $v = 1/(2t + 1)$ (m/s) for $0 \le t \le 8$; $n = 4$

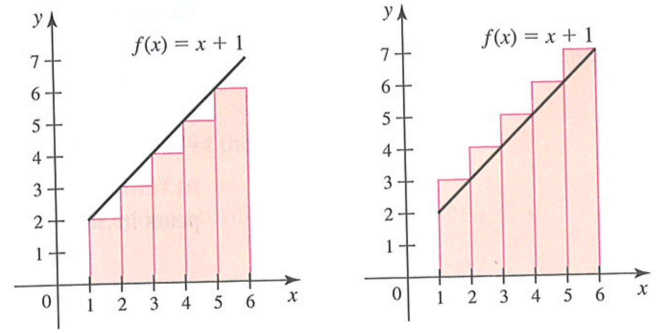
12. $v = t^2/2 + 4$ (ft/s) for $0 \le t \le 12$; $n = 6$

13. $v = 4\sqrt{t + 1}$ (mi/hr) for $0 \le t \le 15$; $n = 5$

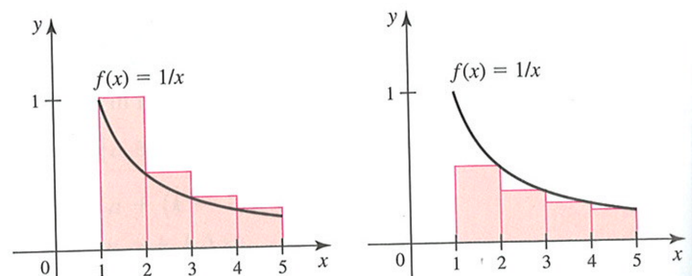
14. $v = (t + 3)/6$ (m/s) for $0 \le t \le 4$; $n = 4$

15–16. Left and right Riemann sums *Use the figures to calculate the left and right Riemann sums for f on the given interval and the given value of n.*

15. $f(x) = x + 1$ on $[1, 6]$; $n = 5$



16. $f(x) = \dfrac{1}{x}$ on $[1, 5]$; $n = 4$



17–20. Left and right Riemann sums *Complete the following steps for the given function, interval, and value of n.*

a. Sketch the graph of the function on the given interval.

b. Calculate Δx and the grid points $x_0, x_1, \ldots, x_n$.

c. Illustrate the left and right Riemann sums, and determine which Riemann sum underestimates and which sum overestimates the area under the curve.

d. Calculate the left and right Riemann sums.

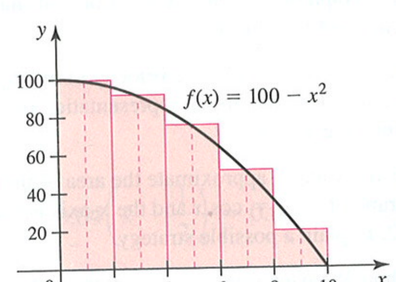
17. $f(x) = x^2 - 1$ on $[2, 4]$; $n = 4$

18. $f(x) = 2x^2$ on $[1, 6]$; $n = 5$

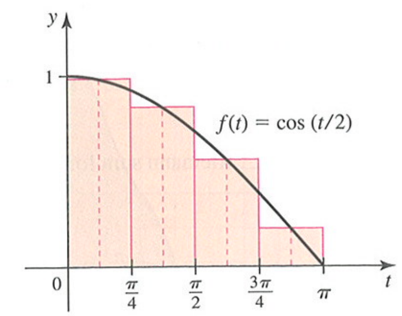
19. $f(x) = \cos x$ on $[0, \pi/2]$; $n = 4$

20. $f(x) = \cos x$ on $[-\pi/2, \pi/2]$; $n = 6$

21. A midpoint Riemann sum Approximate the area of the region bounded by the graph of $f(x) = 100 - x^2$ and the x-axis on $[0, 10]$ with $n = 5$ subintervals. Use the midpoint of each subinterval to determine the height of each rectangle (see figure).



22. A midpoint Riemann sum Approximate the area of the region bounded by the graph of $f(t) = \cos(t/2)$ and the t-axis on $[0, \pi]$ with $n = 4$ subintervals. Use the midpoint of each subinterval to determine the height of each rectangle (see figure).



23–26. Midpoint Riemann sums *Complete the following steps for the given function, interval, and value of n.*

a. Sketch the graph of the function on the given interval.

b. Calculate Δx and the grid points $x_0, x_1, \ldots, x_n$.

c. Illustrate the midpoint Riemann sum by sketching the appropriate rectangles.

d. Calculate the midpoint Riemann sum.

23. $f(x) = \sqrt{x}$ on $[1, 3]$; $n = 4$

24. $f(x) = x^2$ on $[0, 4]$; $n = 4$

25. $f(x) = \dfrac{1}{x}$ on $[1, 6]$; $n = 5$

26. $f(x) = 4 - x$ on $[-1, 4]$; $n = 5$

27–28. Riemann sums from tables *Use the tabulated values of f to evaluate the left and right Riemann sums for the given value of n.*

27. $n = 4$; $[0, 2]$

x	0	0.5	1	1.5	2
$f(x)$	5	3	2	1	1

28. $n = 8$; $[1, 5]$

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$f(x)$	0	2	3	2	2	1	0	2	3

29. Displacement from a table of velocities The velocities (in mi/hr) of an automobile moving along a straight highway over a 2-hr period $0 \le t \le 2$ are given in the following table.

t (hr)	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
v (mi/hr)	50	50	60	60	55	65	50	60	70

a. Sketch a smooth curve passing through the data points.

b. Find the midpoint Riemann sum approximation to the displacement on $[0, 2]$ with $n = 2$ and $n = 4$.

30. Displacement from a table of velocities The velocities (in m/s) of an automobile moving along a straight freeway over a 4-s period $0 \le t \le 4$ are given in the following table.

t (s)	0	0.5	1	1.5	2	2.5	3	3.5	4
v (m/s)	20	25	30	35	30	30	35	40	40

a. Sketch a smooth curve passing through the data points.

b. Find the midpoint Riemann sum approximation to the displacement on $[0, 4]$ with $n = 2$ and $n = 4$ subintervals.

31. Sigma notation Express the following sums using sigma notation. (Answers are not unique.)

a. $1 + 2 + 3 + 4 + 5$

b. $4 + 5 + 6 + 7 + 8 + 9$

c. $1^2 + 2^2 + 3^2 + 4^2$

d. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

32. Sigma notation Express the following sums using sigma notation. (Answers are not unique.)

a. $1 + 3 + 5 + 7 + \cdots + 99$

b. $4 + 9 + 14 + \cdots + 44$

c. $3 + 8 + 13 + \cdots + 63$

d. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{49 \cdot 50}$

33. Sigma notation Evaluate the following expressions.

a. $\displaystyle\sum_{k=1}^{10} k$

b. $\displaystyle\sum_{k=1}^{6} (2k + 1)$

c. $\displaystyle\sum_{k=1}^{4} k^2$

d. $\displaystyle\sum_{n=1}^{5} (1 + n^2)$

e. $\displaystyle\sum_{m=1}^{3} \frac{2m + 2}{3}$

f. $\displaystyle\sum_{j=1}^{3} (3j - 4)$

g. $\displaystyle\sum_{p=1}^{5} (2p + p^2)$

h. $\displaystyle\sum_{n=0}^{4} \sin \frac{n\pi}{2}$

34. Evaluating sums Evaluate the following expressions by two methods.

(i) Use Theorem 5.1. (ii) Use a calculator.

a. $\displaystyle\sum_{k=1}^{45} k$

b. $\displaystyle\sum_{k=1}^{45} (5k - 1)$

c. $\displaystyle\sum_{k=1}^{75} 2k^2$

d. $\displaystyle\sum_{n=1}^{50} (1 + n^2)$

e. $\displaystyle\sum_{m=1}^{75} \frac{2m + 2}{3}$

f. $\displaystyle\sum_{j=1}^{20} (3j - 4)$

g. $\displaystyle\sum_{p=1}^{35} (2p + p^2)$

h. $\displaystyle\sum_{n=0}^{40} (n^2 + 3n - 1)$

35–38. Riemann sums for larger values of *n* *Complete the following steps for the given function f and interval.*

a. For the given value of n, use sigma notation to write the left, right, and midpoint Riemann sums. Then evaluate each sum using a calculator.

b. Based on the approximations found in part (a), estimate the area of the region bounded by the graph of f on the interval.

35. $f(x) = \sqrt{x}$ for $[0, 4]$; $n = 40$

36. $f(x) = x^2 + 1$ for $[-1, 1]$; $n = 50$

37. $f(x) = x^2 - 1$ for $[2, 7]$; $n = 75$

38. $f(x) = \cos 2x$ for $[0, \pi/4]$; $n = 60$

39–44. Approximating areas with a calculator *Use a calculator and right Riemann sums to approximate the area of the region described. Present your calculations in a table showing the approximations for $n = 10, 30, 60,$ and 80 subintervals. Comment on whether your approximations appear to approach a limit.*

39. The region bounded by the graph of $f(x) = 4 - x^2$ and the x-axis on the interval $[-2, 2]$
40. The region bounded by the graph of $f(x) = x^2 + 1$ and the x-axis on the interval $[0, 2]$
41. The region bounded by the graph of $f(x) = 2 - 2\sin x$ and the x-axis on the interval $[-\pi/2, \pi/2]$
42. The region bounded by the graph of $f(x) = 2^x$ and the x-axis on the interval $[1, 2]$
43. The region bounded by the graph of $f(x) = \ln x$ and the x-axis on the interval $[1, e]$
44. The region bounded by the graph of $f(x) = \sqrt{x + 1}$ and the x-axis on the interval $[0, 3]$

Further Explorations

45. **Explain why or why not** State whether the following statements are true and give an explanation or counterexample.
    a. Consider the linear function $f(x) = 2x + 5$ and the region bounded by its graph and the x-axis on the interval $[3, 6]$. Suppose the area of this region is approximated using midpoint Riemann sums. Then the approximations give the exact area of the region for any number of subintervals.
    b. A left Riemann sum always overestimates the area of a region bounded by a positive increasing function and the x-axis on an interval $[a, b]$.
    c. For an increasing or decreasing nonconstant function and a given value of n on an interval $[a, b]$, the value of the midpoint Riemann sum always lies between the values of the left and right Riemann sums.

46–47. Riemann sums *Evaluate the Riemann sum for f on the given interval for the given values of n and $\bar{x}_k$. Sketch the graph of f and the rectangles used in the Riemann sum.*

46. $f(x) = x^2 + 2$ for $[0, 2]$; $n = 2$; $\bar{x}_1 = 0.25$ and $\bar{x}_2 = 1.75$
47. $f(x) = 1/x$ for $[1, 3]$; $n = 5$; $\bar{x}_1 = 1.1, \bar{x}_2 = 1.5, \bar{x}_3 = 2,$ $\bar{x}_4 = 2.3,$ and $\bar{x}_5 = 3$

48. **Riemann sums for a semicircle** Let $f(x) = \sqrt{1 - x^2}$.
    a. Show that the graph of f is the upper half of a circle of radius 1 centered at the origin.
    b. Estimate the area between the graph of f and the x-axis on the interval $[-1, 1]$ using a midpoint Riemann sum with $n = 25$.
    c. Repeat part (b) using $n = 75$ rectangles.
    d. What happens to the midpoint Riemann sums on $[-1, 1]$ as $n \to \infty$?

49–52. Sigma notation for Riemann sums *Use sigma notation to write the following Riemann sums. Then, evaluate each Riemann sum using Theorem 5.1 or a calculator.*

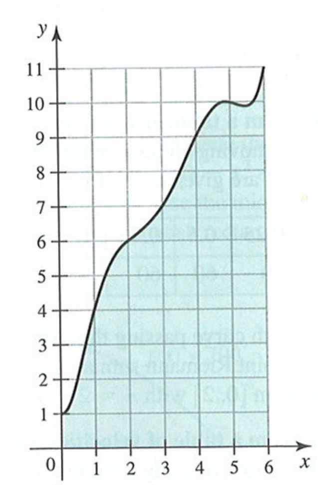
49. The right Riemann sum for $f(x) = x + 1$ on $[0, 4]$ with $n = 50$
50. The left Riemann sum for $f(x) = e^x$ on $[0, \ln 2]$ with $n = 40$
51. The midpoint Riemann sum for $f(x) = x^3$ on $[3, 11]$ with $n = 32$
52. The midpoint Riemann sum for $f(x) = 1 + \cos(\pi x)$ on $[0, 2]$ with $n = 50$

53–56. Identifying Riemann sums *Fill in the blanks with right, left, or midpoint; an interval; and a value of n. In some cases, more than one answer may work.*

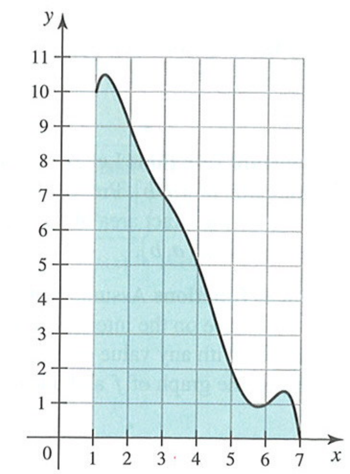
53. $\sum_{k=1}^{4} f(1 + k) \cdot 1$ is a ____ Riemann sum for f on the interval [__, __] with $n =$ ____
54. $\sum_{k=1}^{4} f(2 + k) \cdot 1$ is a ____ Riemann sum for f on the interval [__, __] with $n =$ ____
55. $\sum_{k=1}^{4} f(1.5 + k) \cdot 1$ is a ____ Riemann sum for f on the interval [__, __] with $n =$ ____
56. $\sum_{k=1}^{8} f\left(1.5 + \frac{k}{2}\right) \cdot \frac{1}{2}$ is a ____ Riemann sum for f on the interval [__, __] with $n =$ ____

57. **Approximating areas** Estimate the area of the region bounded by the graph of $f(x) = x^2 + 2$ and the x-axis on $[0, 2]$ in the following ways.
    a. Divide $[0, 2]$ into $n = 4$ subintervals and approximate the area of the region using a left Riemann sum. Illustrate the solution geometrically.
    b. Divide $[0, 2]$ into $n = 4$ subintervals and approximate the area of the region using a midpoint Riemann sum. Illustrate the solution geometrically.
    c. Divide $[0, 2]$ into $n = 4$ subintervals and approximate the area of the region using a right Riemann sum. Illustrate the solution geometrically.

58. **Approximating area from a graph** Approximate the area of the region bounded by the graph (see figure) and the x-axis by dividing the interval $[0, 6]$ into $n = 3$ subintervals. Then use left and right Riemann sums to obtain two different approximations.

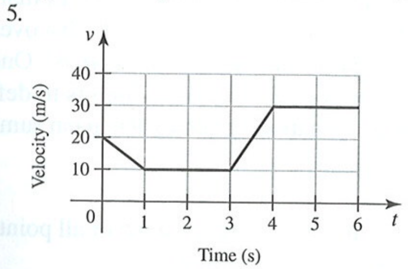


59. **Approximating area from a graph** Approximate the area of the region under the graph (see figure) by dividing the interval $[1, 7]$ into $n = 6$ subintervals. Then use left and right Riemann sums to obtain two different approximations.

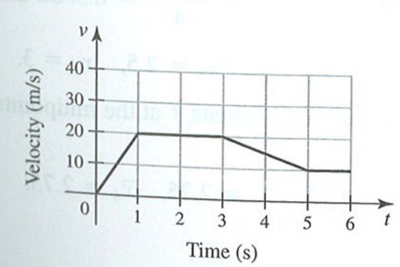


Applications

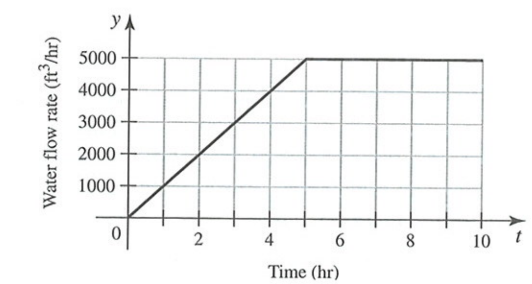
60. **Displacement from a velocity graph** Consider the velocity function for an object moving along a line shown in the figure.
    a. Describe the motion of the object over the interval $[0, 6]$.
    b. Use geometry to find the displacement of the object between $t = 0$ and $t = 3$.
    c. Use geometry to find the displacement of the object between $t = 3$ and $t = 5$.
    d. Assuming that the velocity remains 30 m/s for $t \geq 4$, find the function that gives the displacement between $t = 0$ and any time $t \geq 5$.



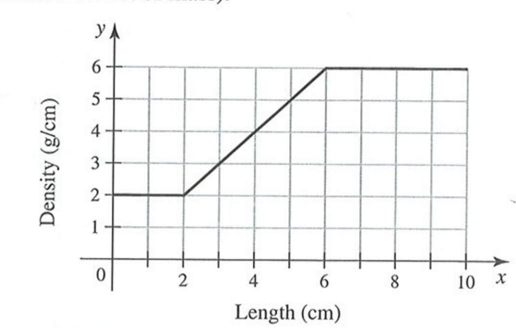
61. **Displacement from a velocity graph** Consider the velocity function for an object moving along a line shown in the figure.
    a. Describe the motion of the object over the interval $[0, 6]$.
    b. Use geometry to find the displacement of the object between $t = 0$ and $t = 2$.
    c. Use geometry to find the displacement of the object between $t = 2$ and $t = 5$.
    d. Assuming that the velocity remains 10 m/s for $t \geq 5$, find the function that gives the displacement between $t = 0$ and any time $t \geq 5$.



62. **Flow rates** Suppose a gauge at the outflow of a reservoir measures the flow rate of water in units of ft^3/hr. In Chapter 6 we show that the total amount of water that flows out of the reservoir is the area under the flow rate curve. Consider the flow rate function shown in the figure.
    a. Find the amount of water (in units of ft^3) that flows out of the reservoir over the interval $[0, 4]$.
    b. Find the amount of water that flows out of the reservoir over the interval $[8, 10]$.
    c. Does more water flow out of the reservoir over the interval $[0, 4]$ or $[4, 6]$?
    d. Show that the units of your answer are consistent with the units of the variables on the axes.

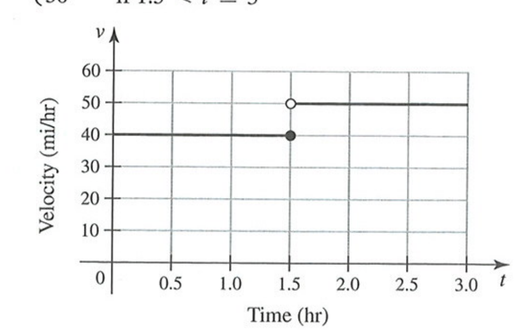


63. **Mass from density** A thin 10-cm rod is made of an alloy whose density varies along its length according to the function shown in the figure. Assume density is measured in units of g/cm. In Chapter 6, we show that the mass of the rod is the area under the density curve.
    a. Find the mass of the left half of the rod ($0 \leq x \leq 5$).
    b. Find the mass of the right half of the rod ($5 \leq x \leq 10$).
    c. Find the mass of the entire rod ($0 \leq x \leq 10$).
    d. Estimate the point along the rod at which it will balance (called the center of mass).

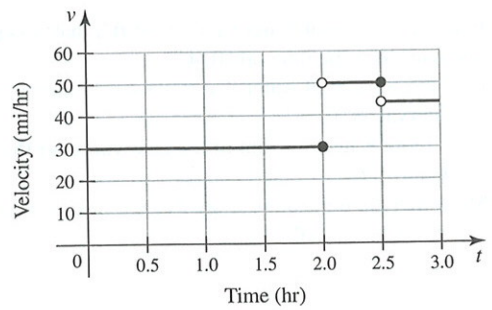


64–65. Displacement from velocity *The following functions describe the velocity of a car (in mi/hr) moving along a straight highway for a 3-hr interval. In each case, find the function that gives the displacement of the car over the interval $[0, t]$, where $0 \leq t \leq 3$.*

64. $v(t) = \begin{cases} 40 & \text{if } 0 \leq t \leq 1.5 \\ 50 & \text{if } 1.5 < t \leq 3 \end{cases}$



65. $v(t) = \begin{cases} 30 & \text{if } 0 \le t \le 2 \\ 50 & \text{if } 2 < t \le 2.5 \\ 44 & \text{if } 2.5 < t \le 3 \end{cases}$



T 66–69. Functions with absolute value *Use a calculator and the method of your choice to approximate the area of the following regions. Present your calculations in a table, showing approximations using $n = 16, 32,$ and 64 subintervals. Comment on whether your approximations appear to approach a limit.*

66. The region bounded by the graph of $f(x) = |25 - x^2|$ and the x-axis on the interval $[0, 10]$

67. The region bounded by the graph of $f(x) = |x(x^2 - 1)|$ and the x-axis on the interval $[-1, 1]$

68. The region bounded by the graph of $f(x) = |\cos 2x|$ and the x-axis on the interval $[0, \pi]$

69. The region bounded by the graph of $f(x) = |1 - x^3|$ and the x-axis on the interval $[-1, 2]$

Additional Exercises

70. Riemann sums for constant functions Let $f(x) = c$, where $c > 0$, be a constant function on $[a, b]$. Prove that any Riemann sum for any value of n gives the exact area of the region between the graph of f and the x-axis on $[a, b]$.

71. Riemann sums for linear functions Assume that the linear function $f(x) = mx + c$ is positive on the interval $[a, b]$. Prove that the midpoint Riemann sum with any value of n gives the exact area of the region between the graph of f and the x-axis on $[a, b]$.

QUICK CHECK ANSWERS

1. 45 mi **2.** 0.25, 0.125, 7.875 **3.** $\Delta x = 2$; $\{1, 3, 5, 7, 9\}$ **4.** The left sum overestimates the area. ◄

5.2 Definite Integrals

We introduced Riemann sums in Section 5.1 as a way to approximate the area of a region bounded by a curve $y = f(x)$ and the x-axis on an interval $[a, b]$. In that discussion, we assumed f to be nonnegative on the interval. Our next task is to discover the geometric meaning of Riemann sums when f is negative on some or all of $[a, b]$. Once this matter is settled, we can proceed to the main event of this section, which is to define the *definite integral*. With definite integrals, the approximations given by Riemann sums become exact.

Net Area

How do we interpret Riemann sums when f is negative at some or all points of $[a, b]$? The answer follows directly from the Riemann sum definition.

EXAMPLE 1 Interpreting Riemann Sums Evaluate and interpret the following Riemann sums for $f(x) = 1 - x^2$ on the interval $[a, b]$ with n equally spaced subintervals.

a. A midpoint Riemann sum with $[a, b] = [1, 3]$ and $n = 4$

b. A left Riemann sum with $[a, b] = [0, 3]$ and $n = 6$

SOLUTION

a. The length of each subinterval is $\Delta x = \dfrac{b - a}{n} = \dfrac{3 - 1}{4} = 0.5$. So the grid points are

$$x_0 = 1, \quad x_1 = 1.5, \quad x_2 = 2, \quad x_3 = 2.5, \quad x_4 = 3.$$

To compute the midpoint Riemann sum, we evaluate f at the midpoints of the subintervals, which are

$$\bar{x}_1 = 1.25, \quad \bar{x}_2 = 1.75, \quad \bar{x}_3 = 2.25, \quad \bar{x}_4 = 2.75.$$

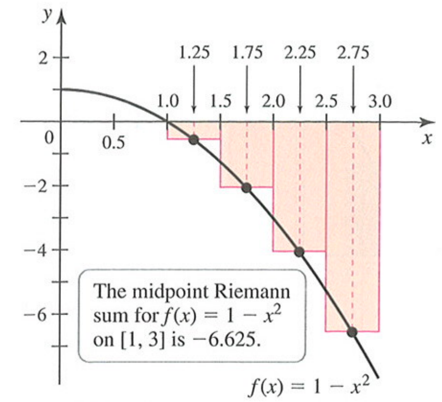


FIGURE 5.16

The resulting midpoint Riemann sum is

$$\begin{aligned}\sum_{k=1}^{n} f(\bar{x}_k)\Delta x &= \sum_{k=1}^{4} f(\bar{x}_k)(0.5) \\ &= f(1.25)(0.5) + f(1.75)(0.5) + f(2.25)(0.5) + f(2.75)(0.5) \\ &= (-0.5625 - 2.0625 - 4.0625 - 6.5625)0.5 \\ &= -6.625.\end{aligned}$$

All values of $f(\bar{x}_k)$ are negative, so the Riemann sum is also negative. Because area is always a nonnegative quantity, this Riemann sum does not approximate an area. Notice, however, that the values of $f(\bar{x}_k)$ are the *negative* of the heights of the corresponding rectangles (Figure 5.16). Therefore, the Riemann sum is an approximation to the *negative* of the area of the region bounded by the curve.

b. The length of each subinterval is $\Delta x = \dfrac{b - a}{n} = \dfrac{3 - 0}{6} = 0.5$ and the grid points are

$$x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 1, \quad x_3 = 1.5, \quad x_4 = 2, \quad x_5 = 2.5, \quad x_6 = 3.$$

To calculate the left Riemann sum, we set $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_6$ equal to the left endpoints of the subintervals:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0.5, \quad \bar{x}_3 = 1, \quad \bar{x}_4 = 1.5, \quad \bar{x}_5 = 2, \quad \bar{x}_6 = 2.5$$

The resulting left Riemann sum is

$$\begin{aligned}\sum_{k=1}^{n} f(\bar{x}_k)\Delta x &= \sum_{k=1}^{6} f(\bar{x}_k)(0.5) \\ &= (\underbrace{f(0) + f(0.5) + f(1)}_{\text{nonnegative contribution}} + \underbrace{f(1.5) + f(2) + f(2.5)}_{\text{negative contribution}})\,0.5 \\ &= (1 + 0.75 + 0 - 1.25 - 3 - 5.25)\,0.5 \\ &= -3.875.\end{aligned}$$

In this case the values of $f(\bar{x}_k)$ are nonnegative for $k = 1, 2,$ and 3 and negative for $k = 4, 5,$ and 6 (Figure 5.17). Where f is positive, we get positive contributions to the Riemann sum and where f is negative, we get negative contributions to the sum.

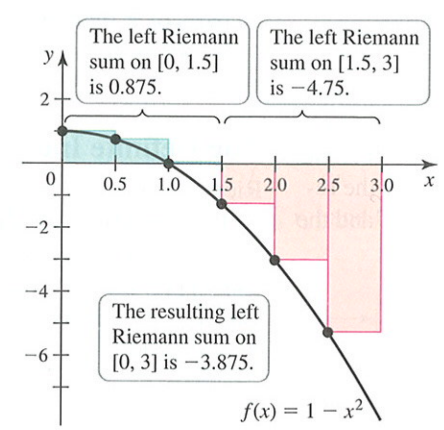


FIGURE 5.17

Related Exercises 11–18 ◄

Let's recap what was learned in Example 1. On intervals where $f(x) < 0$, Riemann sums approximate the *negative* of the area of the region bounded by the curve (Figure 5.18).